Trade-off between efficiency and power of heat engines

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N. Shiraishi and K. Saito, arXiv:1806.09389 (today’s arXiv!)
Part I
Trade-off inequality between entropy production and heat flux
Background
Efficiency and power: key quantities

Efficiency: How much heat is extracted as work.
Efficiency and power: key quantities

Efficiency: How much heat is extracted as work.

Power: How much work is extracted per unit time.
Efficiency and power: key quantities

Efficiency: How much heat is extracted as work.

Power: How much work is extracted per unit time.

What relation holds between efficiency and power?
What general properties we know?

Basic problem
Does a finite power engine attain Carnot efficiency?
What general properties we know?

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Does a finite power engine attain Carnot efficiency?

Even such a simple problem has been open problem!
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Does a finite power engine attain Carnot efficiency?

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Thermodynamics
No restriction.
What general properties we know?

Basic problem
Does a finite power engine attain Carnot efficiency?

Even such a simple problem has been open problem!

Thermodynamics
No restriction.

Liner irreversible thermodynamics
Not prohibited even in linear response regime when time-reversal symmetry is broken.

(G. Benenti, K. Saito, and G. Casati, PRL 106, 230602 (2011).)
Concrete models (in linear regime)...

With magnetic field

B. Sothmann, R. Sanchez, and A. Jordan, EPL 107, 47003 (2014).

Time asymmetric operation

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**Time asymmetric operation**


Within these models in linear regime, Carnot efficiency implies zero power!
Present situation

- General frameworks do not prohibit an engine at CE with finite power.

- **In analyses on concrete models in linear regime,** all models do not attain CE with finite power.

- General trade-off relation between power and efficiency has completely been missing.
Main result
Setup

Assumption

- Dynamics of the engine is described by classical Markov process
- Canonical distribution is invariant

Remarks

- Transient process $\rightarrow$ OK
- Broken time-reversal symmetry $\rightarrow$ OK
- Nonlinear regime $\rightarrow$ OK
Main result (Inequality between heat flux entropy production)
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Key quantities

$J$: heat flux between bath and engine (in general, flux of conserved quantities)

$\sigma$: entropy production rate
Main result (Inequality between heat flux entropy production)

**Key quantities**

\( J \): heat flux between bath and engine (in general, flux of conserved quantities)

\( \sigma \): entropy production rate

Then, the following relation holds (case of single bath)

\[ |J| \leq \sqrt{\Theta \sigma} \]

(\( \Theta \): coefficient depending on state defined later)

(N. Shiraishi, K. Saito, and H. Tasaki, PRL 117, 190601 (2016))
Definition of $\Theta$ dependent on the conditions

$$|J| \leq \sqrt{\Theta \sigma}$$

$\Theta = \Theta^{(1)} :$ General case, but weak a little
(e.g., systems with thermal wall)

$\Theta = \Theta^{(2)} :$ Case with detailed balance, but strong
(e.g., linear Langevin systems, discrete systems without magnetic field)
Power and efficiency (schematics)

Cyclic process with two baths
There must exist isothermal processes, and they possess inevitable dissipation.
Main result (Inequality between power and efficiency)

Cyclic process with two baths, work $W$ and efficiency $\eta$ satisfies

$$\frac{W}{\tau} \leq \bar{\Theta} \beta L \eta (\eta_C - \eta)$$

$\tau$ : cyclic time interval
$\bar{\Theta}$ : average of $\Theta$ (defined later)
$\beta L$ : inverse temperature of cold bath
$\eta_C$ : Carnot efficiency
Main result (Inequality between power and efficiency)

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(N. Shiraishi, K. Saito, and H. Tasaki, PRL 117, 190601 (2016))
Outline of this part

We first consider the case of single heat bath.
- general case ($\Theta = \Theta^{(1)}$)
- with detailed-balance ($\Theta = \Theta^{(2)}$)

We then consider the case of
- single particle underdamped Langevin system
- many-particle systems
- multi-bath

We finally derive trade-off inequality between efficiency and power
Dynamics

Master eq.

\[ \frac{d}{dt} P_{w,t} = \sum_{w'} R_{ww'} P_{w',t} \]

- \( w \): state of engine
- \( R_{ww'} \): transition matrix with \( w' \rightarrow w \)

Normalization condition: \( \sum_w R_{ww'} = 0 \)
## Dynamics

**Master eq.**

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\frac{d}{dt} P_{w,t} = \sum_{w'} R_{ww'} P_{w',t}
\]

- \(w\) : state of engine
- \(R_{ww'}\) : transition matrix with \(w' \rightarrow w\)

Normalization condition: \(\sum_w R_{ww'} = 0\)

We define dual matrix as \(\tilde{R}_{ww'} := e^{-\beta(E_w - E_{w'})} R_{w',w}\).
Definitions of key quantities

Heat flux

\[ J = - \sum_{w'} E_{w'} \frac{dP_{w'}}{dt} \]
Definitions of key quantities

Heat flux

\[ J = - \sum_{w'} E_{w'} \frac{dP_{w'}}{dt} \]

\[ = - \sum_{w,w'} E_{w'} R_{w,w} P_w \]
Definitions of key quantities

Heat flux

\[ J = - \sum_{w'} E_{w'} \frac{dP_{w'}}{dt} \]

\[ = - \sum_{w,w'} E_{w'} R_{w,w} P_w \]

\[ = - \sum_{w,w'} \Delta E_{w'} (R_{w,w} P_w - \tilde{R}_{ww'} P_{w'}) \]

(Energy fluctuation: \( \Delta E_{w'} := E_{w'} - \langle E \rangle \))
Definitions of key quantities

Entropy production rate

\[ \sigma = - \sum_{w} \beta E_w \frac{dP_w}{dt} + \frac{d}{dt} \left( - \sum_{w} P_w \ln P_w \right) \]

- Entropy increase of bath \((dQ/T)\)
- (Shannon) entropy increase of system
Definitions of key quantities

Entropy production rate

\[ \sigma = - \sum_w \beta E_w \frac{dP_w}{dt} + \frac{d}{dt} \left( - \sum_w P_w \ln P_w \right) \]

\[ = \sum_{w,w'} R_{w',w} P_w \ln \frac{R_{w',w} P_w}{\tilde{R}_{ww'} P_{w'}} \]
Lemma: Inequality for relative entropy

For $\sum_x p_x = \sum_x q_x$, relative entropy satisfies

$$D(p_x || q_x) := \sum_x p_x \ln \frac{p_x}{q_x}$$
Lemma: Inequality for relative entropy

For $\sum_x p_x = \sum_x q_x$, relative entropy satisfies

$$D(p_x || q_x) := \sum_x p_x \ln \frac{p_x}{q_x}$$

$$= \sum_x p_x \ln \frac{p_x}{q_x} + q_x - p_x$$
Lemma: Inequality for relative entropy

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$$= \sum_x p_x \ln \frac{p_x}{q_x} + q_x - p_x$$

$$\geq \sum_x \frac{c_0 (p_x - q_x)^2}{p_x + q_x} \quad \left( c_0 = \frac{8}{9} \right)$$
Derivation of main result

\[ |J|^2 = \left| \sum_{w \neq w'} \Delta E_{w'} (R_{w'\,w} P_w - \tilde{R}_{ww'} P_{w'}) \right|^2 \]
Derivation of main result

\[ |J|^2 = \left| \sum_{w \neq w'} \Delta E_{w'} \left( R_{w'w} P_w - \tilde{R}_{ww'} P_{w'} \right) \right|^2 \]

\[ = \left| \sum_{w \neq w'} \Delta E_{w'} \sqrt{R_{w'w} P_w + \tilde{R}_{ww'} P_{w'}} \frac{R_{w'w} P_w - \tilde{R}_{ww'} P_{w'}}{\sqrt{R_{w'w} P_w + \tilde{R}_{ww'} P_{w'}}} \right|^2 \]
Derivation of main result

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|J|^2 = \left| \sum_{w \neq w'} \Delta E_{w'} \left( R_{w',w} P_w - \tilde{R}_{ww'} P_{w'} \right) \right|^2 \\
= \left| \sum_{w \neq w'} \Delta E_{w'} \sqrt{R_{w',w} P_w + \tilde{R}_{ww'} P_{w'}} \frac{R_{w',w} P_w - \tilde{R}_{ww'} P_{w'}}{\sqrt{R_{w',w} P_w + \tilde{R}_{ww'} P_{w'}}} \right|^2 \\
\leq \sum_{w \neq w'} \Delta E_{w'}^2 \left( R_{w',w} P_w + \tilde{R}_{ww'} P_{w'} \right) \cdot \sum_{w \neq w'} \frac{\left( R_{w',w} P_w - \tilde{R}_{ww'} P_{w'} \right)^2}{R_{w',w} P_w + \tilde{R}_{ww'} P_{w'}} \]

Schwarz inequality is used
Derivation of main result

\[ |J|^2 \]

\[ = \left| \sum_{w \neq w'} \Delta E_w' \left( R_{w'w} P_w - \tilde{R}_{ww'} P_w' \right) \right|^2 \]

\[ = \left| \sum_{w \neq w'} \Delta E_w' \sqrt{R_{w'w} P_w + \tilde{R}_{ww'} P_w'} \frac{R_{w'w} P_w - \tilde{R}_{ww'} P_w'}{\sqrt{R_{w'w} P_w + \tilde{R}_{ww'} P_w'}} \right|^2 \]

\[ \leq \sum_{w \neq w'} \Delta E^2_w \left( R_{w'w} P_w + \tilde{R}_{ww'} P_w' \right) \cdot \sum_{w \neq w'} \frac{(R_{w'w} P_w - \tilde{R}_{ww'} P_w')^2}{R_{w'w} P_w + \tilde{R}_{ww'} P_w'} \]

\[ \leq \frac{1}{c_0} \sum_{w \neq w'} \Delta E^2_w \left( R_{w'w} P_w + \tilde{R}_{ww'} P_w' \right) \sum_{w \neq w'} R_{w'w} P(w) \ln \frac{R_{w'w} P(w)}{\tilde{R}_{ww'} P(w')} \]
Derivation of main result

\[ |J|^2 \]

\[ = \left| \sum_{w \neq w'} \Delta E_{w'}(R_{w'\,w}P_w - \tilde{R}_{w\,w'}P_{w'}) \right|^2 \]

\[ = \sum_{w \neq w'} \Delta E_{w'} \left( \frac{R_{w'\,w}P_w - \tilde{R}_{w\,w'}P_{w'}}{\sqrt{R_{w'\,w}P_w + \tilde{R}_{w\,w'}P_{w'}}} \right)^2 \]

\[ \leq \sum_{w \neq w'} \Delta E_{w'}^2 \left( R_{w'\,w}P_w + \tilde{R}_{w\,w'}P_{w'} \right) \sum_{w \neq w'} \left( \frac{R_{w'\,w}P_w - \tilde{R}_{w\,w'}P_{w'}}{R_{w'\,w}P_w + \tilde{R}_{w\,w'}P_{w'}} \right)^2 \]

\[ \leq \frac{1}{c_0} \sum_{w \neq w'} \Delta E_{w'}^2 \left( R_{w'\,w}P_w + \tilde{R}_{w\,w'}P_{w'} \right) \sum_{w \neq w'} R_{w'\,w}P(w) \ln \frac{R_{w'\,w}P(w)}{\tilde{R}_{w\,w'}P(w')} \]

\[ = \Theta \sigma \]
Inequality between heat flux and entropy production rate (general)

\[ |\mathbf{J}| \leq \sqrt{\Theta^{(1)} \sigma} \]

\[ \Theta^{(1)} := \frac{9}{8} \sum_{w \neq w'} \Delta E_{w'}^2 \left( R_{w'w} P_w + R_{ww'} P_{w'} \right) \]

(We used
\[ \sum_{w(\neq w')} \tilde{R}_{ww'} = -\tilde{R}_{w'w} = -R_{w'w} = \sum_{w(\neq w')} R_{ww'} \])
Case with detailed balance

Detailed balance

\[
\frac{R_{ww'}}{R_{w'w}} = e^{-\beta (E_w - E_{w'})}
\]

Remark: we do NOT take time reversal of \( w \).

This is different from

\[
\frac{R_{ww'}}{R^\dagger \w'\w'^*} = e^{-\beta (E_w - E_{w'})}
\]
Rewrite $J$ and $\sigma$

In this case, $\tilde{R}_{ww'} := e^{-\beta(E_w - E_{w'})} R_{w'w} = R_{ww'}$
Rewrite $J$ and $\sigma$

In this case, $\tilde{R}_{ww'} := e^{-\beta (E_w - E_{w'})} R_{w'w} = R_{ww'}$

$$J = - \sum_{w,w'} E_{w'} (R_{w'w} P_w - R_{ww'} P_{w'})$$

$$= - \frac{1}{2} \sum_{w,w'} (E_{w'} - E_w) (R_{w'w} P_w - R_{ww'} P_{w'})$$

(cf: $J = - \sum_{w,w'} \Delta E_{w'} (R_{w'w} P_w - \tilde{R}_{ww'} P_{w'})$)
Rewrite $J$ and $\sigma$

\[ \sigma = \sum_{w \neq w'} R_{w', w} P_w \ln \frac{R_{w', w} P_w}{R_{ww', w} P_{w'}} \]
Rewrite $J$ and $\sigma$

\[
\sigma = \sum_{w \neq w'} R_{w', w} P_w \ln \frac{R_{w', w} P_w}{R_{ww', P_w'}}
\]

\[
= \frac{1}{2} \sum_{w \neq w'} (R_{w', w} P_w - R_{ww', P_w'}) \ln \frac{R_{w', w} P_w}{R_{ww', P_w'}}
\]
Rewrite $J$ and $\sigma$

\[ \sigma = \sum_{w \neq w'} R_{w'w} P_w \ln \frac{R_{w'w} P_w}{R_{ww'} P_{w'}}, \]

\[ = \frac{1}{2} \sum_{w \neq w'} (R_{w'w} P_w - R_{ww'} P_{w'}) \ln \frac{R_{w'w} P_w}{R_{ww'} P_{w'}}, \]

\[ \geq \frac{1}{2} \sum_{w \neq w'} \frac{2(R_{w'w} P_w - R_{ww'} P_{w'})^2}{R_{w'w} P_w + R_{ww'} P_{w'}}. \]
Rewrite $J$ and $\sigma$

\[
\sigma = \sum_{w \neq w'} R_{w',w} P_w \ln \frac{R_{w',w} P_w}{R_{ww',w} P_{w'}}
\]

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\]

\[
\geq \frac{1}{2} \sum_{w \neq w'} \frac{2(R_{w',w} P_w - R_{ww',w} P_{w'})^2}{R_{w',w} P_w + R_{ww',w} P_{w'}}
\]

(cf: $\sigma \geq \sum_{w \neq w'} \frac{c_0 (R_{w',w} P_w - \bar{R}_{w',w} P_{w'})^2}{R_{w',w} P_w + \bar{R}_{w',w} P_{w'}}$)
Inequality between heat flux and entropy production rate (strong)

\[ |J| \leq \sqrt{\Theta^{(2)}} \sigma \]

\[ \Theta^{(2)} := \frac{1}{2} \sum_{w \neq w'} (E_{w'} - E_w)^2 R_{w'w} P_w \]

(cf: \( \Theta^{(1)} := \frac{9}{8} \sum_{w \neq w'} \Delta E_{w'}^2, (R_{w'w} P_w + R_{ww'} P_{w'}) \))
General properties of $\Theta$

- $\Theta^{(2)} = \langle \frac{\gamma |p|^2}{\beta m^2} \rangle$ for underdamped Langevin systems.

- $\Theta^{(2)} = \langle \frac{\gamma F(x)^2}{\beta} \rangle$ for overdamped Langevin systems.

- In linear regime, $\Theta^{(2)} = \kappa$ (thermal conductivity) and equality holds ($|J| = \sqrt{\Theta \sigma}$).

- $\Theta^{(2)}$ is “the second moment of instant heat flux”
Application to many-body and multi-bath systems with Hamilton dynamics

Total transition rate is decomposed into

\[ R_{ww'} = R_{ww'}^0 + \sum_{\nu} \sum_{i} R_{ww'}^{\nu,i} \]
Application to many-body and multi-bath systems with Hamilton dynamics

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\[ R_{ww'} = R_{ww'}^0 + \sum_{\nu} \sum_{i} R_{ww'}^{\nu,i} \]

Hamiltonian dynamics
Application to many-body and multi-bath systems with Hamilton dynamics

Total transition rate is decomposed into

\[ R_{ww'} = R^0_{ww'} + \sum_{\nu} \sum_{i} R^{\nu,i}_{ww'} \]

Hamiltonian dynamics

- \( R_{ww'} \) is the total transition rate.
- \( R^0_{ww'} \) represents the Hamiltonian dynamics.
- The sum over \( \nu \) labels the bath, and the sum over \( i \) labels the particle.

Diagram:
- A red box labeled \( \nu \)-th bath with green circles (representing particles).
- A line indicating the interaction between the bath and the particle.
Hamilton and stochastic parts in underdamped Langevin system

\[ L := -v \frac{\partial}{\partial x} + \frac{\partial}{\partial v} \left( \gamma v + \frac{1}{m} \frac{dU}{dx} + B \times v \right) + \frac{\gamma}{\beta m} \frac{\partial^2}{\partial v^2} \]

\[ \frac{d}{dt} P_{x,v} = LP_{x,v} \]
Hamilton and stochastic parts in underdamped Langevin system

\[
L := -v \frac{\partial}{\partial x} + \frac{\partial}{\partial v} \cdot \left( \gamma v + \frac{1}{m} \frac{dU}{dx} + B \times v \right) + \frac{\gamma}{\beta m} \frac{\partial^2}{\partial v^2}
\]

\[
L^0 := -v \frac{\partial}{\partial x} + \frac{\partial}{\partial v} \cdot \left( \frac{1}{m} \frac{dU}{dx} + B \times v \right)
\]

\[
L^{1,1} := \frac{\gamma}{m} \left\{ \frac{\partial}{\partial v} \cdot v + \frac{1}{\beta m} \frac{\partial^2}{\partial v^2} \right\}
\]
Properties of \( R^0 \ (L^0) \) and \( R^{\nu,i} \ (L^{1,1}) \)

\[
L^0 := -v \frac{\partial}{\partial x} + \frac{\partial}{\partial v} \cdot \left( \frac{1}{m} \frac{dU}{dx} + B \times v \right)
\]
Properties of $R^0 (L^0)$ and $R^{\nu,i} (L^{1,1})$

$L^0 := -v \frac{\partial}{\partial x} + \frac{\partial}{\partial v} \cdot \left( \frac{1}{m} \frac{dU}{dx} + B \times v \right)$

- Effects of magnetic field, many-body interactions, inertia is taken in $R^0 (L^0)$.
- $R^0 (L^0)$ conserves both energy and entropy.
  $\rightarrow$ $R^0 (L^0)$ is irrelevant to $J$ and $\sigma$!
Properties of $R^0 (L^0)$ and $R^{\nu,i} (L^{1,1})$

\[ L^0 := -v \frac{\partial}{\partial x} + \frac{\partial}{\partial v} \cdot \left( \frac{1}{m} \frac{dU}{dx} + B \times v \right) \]

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\[ L^{1,1} := \gamma \left\{ \frac{\partial}{\partial v} \cdot v + \frac{1}{\beta m} \frac{\partial^2}{\partial v^2} \right\} \]
Properties of $R^0 \ (L^0)$ and $R^{\nu,i} \ (L^{1,1})$

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L^0 := -v \frac{\partial}{\partial x} + \frac{\partial}{\partial v} \cdot \left( \frac{1}{m} \frac{dU}{dx} + B \times v \right)
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- $R^0 \ (L^0)$ conserves both energy and entropy.
  \[\rightarrow R^0 \ (L^0) \text{ is irrelevant to } J \text{ and } \sigma!\]

\[
L^{1,1} := \frac{\gamma}{m} \left\{ \frac{\partial}{\partial v} \cdot v + \frac{1}{\beta m} \frac{\partial^2}{\partial v^2} \right\}
\]

- $R^{\nu,i} \ (L^{1,1})$ acts only on velocity $\nu$, not position $x$. 
Properties of $R^{\nu,i} (L^{1,1})$ in linear Langevin systems

For linear Langevin systems, corresponding transition rate $R^{1,1}_{\nu'\nu}$ satisfies $R^{1,1}_{\nu'\nu} = R^{1,1}_{-\nu' -\nu}$, which implies detailed balance.
Properties of $R^{\nu,i} \left( L^{1,1} \right)$ in linear Langevin systems

$$L^{1,1} := \frac{\gamma}{m} \left\{ \frac{\partial}{\partial \nu} \cdot \nu + \frac{1}{\beta m} \frac{\partial^2}{\partial \nu^2} \right\}$$

For linear Langevin systems, corresponding transition rate $R_{\nu',\nu}^{1,1}$ satisfies $R_{\nu',\nu}^{1,1} = R_{-\nu',-\nu}^{1,1}$, which implies detailed balance.

This reflect spatial symmetry of noise!
$\Theta$ in multi-particle case and thermodynamic limit

$$|\mathcal{H}| = \sum_i |J_i| \leq \sum_i \sqrt{\Theta_i \sigma_i}$$

$i$-th particle
\[ |V| = \sum_i |J_i| \leq \sum_i \sqrt{\Theta_i \sigma_i} \]

\[ \leq \sqrt{\left( \sum_i \Theta_i \right) \left( \sum_i \sigma_i \right)} \]

\( \Theta \) in multi-particle case and thermodynamic limit
Θ in multi-particle case and thermodynamic limit

\[ |J| = \sum_{i} |J_i| \leq \sum_{i} \sqrt{\Theta_i \sigma_i} \]

\[ \leq \sqrt{\left( \sum_{i} \Theta_i \right) \left( \sum_{i} \sigma_i \right)} \]

\[ = \sqrt{\Theta \sigma} \]
$\Theta$ in multi-particle case and thermodynamic limit

$$|J| = \sum_{i} |J_i| \leq \sum_{i} \sqrt{\Theta_i \sigma_i}$$

$$\leq \sqrt{\left(\sum_{i} \Theta_i\right) \left(\sum_{i} \sigma_i\right)}$$

$$= \sqrt{\Theta \sigma}$$

All of $J, \sigma, \Theta := \sum_{i} \Theta_i$ are proportional to volume $V \rightarrow |J| \leq \sqrt{\Theta \sigma}$ is meaningful even in $V \rightarrow \infty$. 
Multi-bath case

In a similar manner, \( \Theta := \sum_{\nu} \Theta_{\nu} \) satisfies

\[
\sum_{\nu} |J_{\nu}| \leq \sqrt{\Theta \sigma}
\]
Derivation of power-efficiency trade-off

Cyclic process with two baths

Thermodynamics leads to

\[ \Delta S = -\beta_H Q_H + \beta_L Q_L \]

\[
\eta(\eta_C - \eta) = \frac{W}{Q_H} \frac{\beta_L Q_L - \beta_H Q_H}{\beta_L Q_H} 
\]

\[ = \frac{W \Delta S}{\beta_L Q_H \beta_L Q_H} \]

Thermodynamics leads to
Time integration of inequality

General inequality \[ \sum_{\nu} |J_{\nu}| \leq \sqrt{\Theta \sigma} \]

By integrating with time, and using Schwarz inequality

\[
\left( \int_{0}^{\tau} dt \sum_{\nu} |J_{\nu}| \right)^2 \leq \left( \int_{0}^{\tau} dt \sqrt{\Theta \sigma} \right)^2 \\
\leq \left( \int_{0}^{\tau} dt \Theta \right) \left( \int_{0}^{\tau} dt \sigma \right) = \tau \Theta \Delta S \\
\left( \Theta := \frac{1}{\tau} \int_{0}^{\tau} dt \Theta \right)
\]
Time integration of inequality

General inequality \[ \sum_{\nu} |J_{\nu}| \leq \sqrt{\Theta \sigma} \]

By integrating with time, and using Schwarz inequality

\[ \left( \int_0^\tau dt \sum_{\nu} |J_{\nu}| \right)^2 \leq \left( \int_0^\tau dt \sqrt{\Theta \sigma} \right)^2 \]

\[ \leq \left( \int_0^\tau dt \Theta \right) \left( \int_0^\tau dt \sigma \right) = \tau \Theta \Delta S \]

\[ (\Theta := \frac{1}{\tau} \int_0^\tau dt \Theta) \]

\[ Q_H = \int dt J_H \text{ etc. leads to } (Q_H + Q_L)^2 \leq \tau \Theta \Delta S \]
Derivation of power-efficiency trade-off

$$\eta(\eta_c - \eta) = \frac{W \Delta S}{\beta L Q_H^2}$$
Derivation of power-efficiency trade-off

\[
\eta (\eta_C - \eta) = \frac{W \Delta S}{\beta L Q_H^2} \frac{W (Q_H + Q_L)^2}{\beta L Q_H^2 \frac{\tau \Theta}{}}
\]
Derivation of power-efficiency trade-off

\[ \eta(\eta_C - \eta) = \frac{W \Delta S}{\beta_L Q_H^2} \]

\[ \geq \frac{W}{\beta_L Q_H^2} \left( \frac{(Q_H + Q_L)^2}{\tau \Theta} \right) \]

\[ \geq \frac{W}{\beta_L} \frac{1}{\tau \Theta} \]
Derivation of power-efficiency trade-off

$$\eta(\eta_C - \eta) = \frac{W \Delta S}{\beta_L Q_H^2}$$

$$\geq \frac{W}{\beta_L Q_H^2} \frac{(Q_H + Q_L)^2}{\tau \Theta}$$

$$\geq \frac{W}{\beta_L \tau \Theta}$$

$$\frac{W}{\tau} \leq \bar{\Theta} \beta_L \eta(\eta_C - \eta)$$
Quantum case and non-Markovian case

• We can extend our result to quantum Markov process by considering microscopic origin of quantum Markov process.

• Trade-off inequality between speed and efficiency for quantum non-Markovian system is derived with completely different approach (employing Lieb-Robinson bound and quantum information geometry)

(N. Shiraishi and H. Tajima, PRE 96, 022138 (2017))
Summary (of this part)

• Trade-off between heat flux and entropy production rate is obtained:
  \[ |J| \leq \sqrt{\Theta \sigma} \]

• Using this, power-efficiency trade-off is obtained:
  \[ \frac{W}{\tau} \leq \bar{\Theta} \beta L \eta (\eta_C - \eta) \]

• As its corollary, we obtain no-go theorem of coexistence of finite power and Carnot efficiency.
Part II
Speed limit for classical systems

Problem: Given Initial and final states (distributions). How quick can we transform this state?
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We can tune how to change the control parameters.
Speed limit: problem

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Problem: Given Initial and final states (distributions). How quick can we transform this state?

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Quantum speed limit

Mandelstam-Tamm relation: \[ \frac{\mathcal{L}(\rho_i, \rho_f)}{\Delta E/h} \leq \tau \]

(L. Mandelstam and I. Tamm, J. Phys. (USSR) 9, 249 (1945))

Energy fluctuation bounds the speed of operation.
(Background: uncertainty relation)
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What about classical systems?
Classical speed limits: some attempts

Some formal extensions to classical Hamiltonian/stochastic systems

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Physical picture/meaning is highly unclear!
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Overdamped Langevin systems

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Overdamped Langevin systems


Physical picture is clear.
But system is very specific!
Setting and goal of this part

**System**: general Markov process on discrete states with detailed-balance condition.

Initial and final distributions \((p \text{ and } p')\) are given.
Setting and goal of this part

System: general Markov process on discrete states with detailed-balance condition.

Initial and final distributions ($p$ and $p'$) are given.

What we want to obtain is...

$$\mathcal{L}(p, p')^2 \leq \tau$$
Setting and goal of this part

**System**: general Markov process on discrete states with detailed-balance condition.

Initial and final distributions \((p)\) and \((p')\) are given.

What we want to obtain is...

\[
\frac{\mathcal{L}(p, p')^2}{\blacksquare \blacktriangle} \leq \tau
\]

established physical quantities
Main result

\[ \frac{\mathcal{L}(p, p')^2}{2\Sigma\langle A \rangle} \leq \tau \]

\( \mathcal{L}(p, p') := \sum_w |p_w - p'_w| : \) total variation distance

\( \Sigma : \) total entropy production

\( \langle A \rangle : \) averaged dynamical activity \( \int_0^\tau dtA(t) \)
What is dynamical activity?

Dynamical activity: How frequently jumps occur.

\[ A(t) := \sum_{w,w'} R_{w',w} p_w(t) \]

Activity determines **time-scale of dynamics**.
What is dynamical activity?

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Activity

\[ +1 \quad +1 \quad +1 \]

cf) Current

\[ +1 \quad -1 \]
What is dynamical activity?

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Activity determines **time-scale of dynamics**.

Physical meaning of our inequality

\[
\frac{\mathcal{L}(p, p')^2}{2\Sigma\langle A \rangle} \leq \tau
\]
Physical meaning of our inequality

Length between initial and final states

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Time-scale of dynamics (cf: Planck constant)
Physical meaning of our inequality

\[ \frac{\mathcal{L}(p, p')^2}{2\Sigma \langle A \rangle} \leq \tau \]

Length between initial and final states

Entropy production: Cost of quick state transformation

Time-scale of dynamics (cf: Planck constant)
Derivation (instantaneous quantities)

$$\sum_{w} \left| \frac{d}{dt} p_w \right|$$
Derivation (instantaneous quantities)

\[
\sum_{w} \left| \frac{d}{dt} p_{w} \right| = \sum_{w} \left| \sum_{w' \neq w} (R_{w'w} p_{w} - R_{ww'} p_{w'}) \right|
\]
Derivation (instantaneous quantities)

\[
\sum_w \left| \frac{d}{dt} p_w \right |
\]

\[
= \sum_w \left| \sum_{w' \neq w} (R_{w'w} p_w - R_{ww'} p_{w'}) \right |
\]

\[
\leq \sum_w \sqrt{\sum_{w' \neq w} (R_{w'w} p_w + R_{ww'} p_{w'}) \cdot \sum_{w' \neq w} \frac{(R_{w'w} p_w - R_{ww'} p_{w'})^2}{R_{w'w} p_w + R_{ww'} p_{w'}}}
\]
Derivation (instantaneous quantities)

\[ \sum \left| \frac{d}{dt} p_w \right| = \sum \left| \sum_{w'} (R'_{w'} p_w - R_{ww'} p'_w) \right| \leq \sum \sqrt{\sum_{w'} (R'_{w'} p_w + R_{ww'} p'_w)} \cdot \sum_{w'} \frac{(R'_{w'} p_w - R_{ww'} p'_w)^2}{R'_{w'} p_w + R_{ww'} p'_w} \leq \sum (R'_{w'} p_w + R_{ww'} p'_w) \cdot \sum_{w'} \frac{(R'_{w'} p_w - R_{ww'} p'_w)^2}{R'_{w'} p_w + R_{ww'} p'_w} \]
Derivation (instantaneous quantities)

\[
\sum \left| \frac{d}{dt} p_w \right| \\
= \sum \left| \sum_{w'} (R'_{w'}P_w - R_{ww'}P_{w'}) \right|
\leq \sum \sqrt{\sum_{w'} (R'_{w'}P_w + R_{ww'}P_{w'}) \cdot \sum_{w'} \frac{(R'_{w'}P_w - R_{ww'}P_{w'})^2}{R'_{w'}P_w + R_{ww'}P_{w'}}}
\leq \sqrt{\sum_{w', \neq w} (R'_{w'}P_w + R_{ww'}P_{w'}) \cdot \sum_{w', \neq w} \frac{(R'_{w'}P_w - R_{ww'}P_{w'})^2}{R'_{w'}P_w + R_{ww'}P_{w'}}}
\leq \sqrt{2A\sigma}
\]
Derivation (time integration)

\[ \mathcal{L}(p_i, p_f) \leq \sum_w \int_0^\tau dt \left| \frac{d}{dt} p_w \right| \]

\[ \leq \int_0^\tau dt \sqrt{2\sigma A} \leq \sqrt{2\tau \Sigma \langle A \rangle} \]

This is the desired result!

\[ \frac{\mathcal{L}(p, p\prime)^2}{2\Sigma \langle A \rangle} \leq \tau \]
Fro systems without detailed-balance condition

Case with detailed-balance condition

\[ \frac{\mathcal{L}(p, p')^2}{2\Sigma\langle A \rangle} \leq \tau \]
Fro systems without detailed-balance condition

Case with detailed-balance condition
\[ \frac{\mathcal{L}(p, p')^2}{2\Sigma\langle A \rangle} \leq \tau \]

Case without detailed-balance condition
\[ \frac{c_0 \mathcal{L}(p, p')^2}{2\Sigma_{HS}\langle A \rangle} \leq \tau \]

\( \Sigma_{HS} \): Hatano-Sasa entropy production

(Heat \( \beta Q_{w \rightarrow w'} \) is replaced by excess heat \( \ln \frac{p^{ss}_{w'}}{p^{ss}_w} \))

Part III
Maximum efficiency for autonomous engines
Maximum efficiency of autonomous engines

Autonomous: stationary & not controlled externally.

Question: When autonomous engines attain the Carnot efficiency?
Maximum efficiency of autonomous engines

Autonomous: stationary & not controlled externally.

Question: When autonomous engines attain the Carnot efficiency?

Feynman ratchet

(J. M. R. Parrondo & P. Espanol, Am. J. Phys. 64. 1125 (1996))
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Thermoelectricity

(J. M. R. Parrondo & P. Espanol, Am. J. Phys. 64. 1125 (1996))


(G. D. Mahan and J. O. Sofo, PNAS 93, 7436 (1996))
What is general necessary condition to attain the Carnot efficiency?

**Known results**

In linear response regime ($\Delta T, \Delta \mu \cong 0$), tight-coupling condition (determinant of Onsager matrix $= 0$) is necessary and sufficient condition.
What is general necessary condition to attain the Carnot efficiency?

**Known results**
In linear response regime \((\Delta T, \Delta \mu \cong 0)\), tight-coupling condition (determinant of Onsager matrix =0) is necessary and sufficient condition.

- What about general \(\Delta T, \Delta \mu > 0\) regime (nonlinear regime)?
- Is there any relation between condition for linear regime and that for nonlinear regime?
Results

• A kind of **singularity** is necessary to attain the Carnot efficiency in nonlinear regime.

• This singularity condition provides different consequences between in **small systems** (finite size systems) and in **macroscopic systems** (systems in thermodynamic limit).

Classification

particle bath ($\mu_1$) \hspace{2cm} work extraction \hspace{2cm} particle bath ($\mu_2$)

particle current

single heat bath
We plot \((\mu_1, \mu_2)\) attaining the Carnot efficiency.
We plot \((\mu_1, \mu_2)\) attaining the Carnot efficiency.
Three classes in small and macroscopic engines

When $\eta_{\text{Carnot}}$ is attainable?

<table>
<thead>
<tr>
<th>Required condition</th>
<th>Only in linear regime</th>
<th>Within specific pair</th>
<th>Always</th>
</tr>
</thead>
<tbody>
<tr>
<td>Small engine</td>
<td>×</td>
<td>×</td>
<td>○</td>
</tr>
<tr>
<td>Macro engine</td>
<td>○</td>
<td>○</td>
<td>○</td>
</tr>
</tbody>
</table>

Summary

• Trade-off between heat flux and entropy production rate is obtained:
  \[ |J| \leq \sqrt{\Theta \sigma} \]

• Using this, power-efficiency trade-off is obtained:
  \[ \frac{W}{\tau} \leq \overline{\Theta} \beta_L \eta (\eta_C - \eta) \]

• Entropy production also bounds the speed of state transformation
  \[ \frac{\mathcal{L}(p,p')^2}{2\Sigma\langle A \rangle} \leq \tau \]
How to move the engine

closed process

equi-$\mu$ process ($\mu_H$)

equi-$\mu$ process ($\mu_L$)

closed process
State space of total system

$\nu = 0$

$\nu = 1/V$

$\nu = 2/V$

$(X, \nu)$ specifies the total state

$X \in \{A\sim D\}$

$\nu$: particle number/volume of A

This description applies to general engines
Problem and approach

Goal: Derive conditions to attain Carnot efficiency.

Suppose that an engine **attains the Carnot efficiency**, and clarify what properties this engine should satisfy.

(Remark: Although explanation is on the autonomous Carnot engine, it applies to general engines)
Consequence from Carnot efficiency: 1

Equilibration condition: States attached to each particle bath are equilibrium distribution.

Entropy production = 0 ⇔ partial entropy production = 0. Its equality condition implies above condition.
Consequence from Carnot efficiency: 2

Carnot efficiency implies zero power.

→We consider presence/absence of dissipation (particle leakage) under the condition that there is no probability flux between A-D, B-C.
Particle leakage between AD

\[ P_{D \to A}(\nu) : \text{particle number distribution (} \nu := \frac{n}{V_A} \text{)} \]
under the condition that \( D \to A \) occurs

For zero particle leakage, the following two condition is necessary.

Condition 1 : \( P_{D \to A}(\nu) \propto \delta(\nu - \nu^*_{D \to A}) \)
Condition 2 : \( \nu^*_{D \to A} = \nu^*_{A \to D} = \nu^* \)
Necessary and sufficient condition for Carnot efficiency

Condition for Carnot efficiency
Transitions between different particle baths satisfy
1. Transition occurs only at a particular $\nu = \nu^*$ (delta function type)
2. The $\nu^*$ for $A \rightarrow D$ and for $D \rightarrow A$ are the same

(Remark: This is also necessary for linear regime)

What system satisfies the above conditions?
Difference between finite systems and macroscopic systems

1. Transition occurs only a particular $\nu = \nu^*$ (delta function type)
2. The $\nu^*$ for $A \rightarrow D$ and for $D \rightarrow A$ are the same

Case of finite systems
1 is nontrivial condition. 1 directly implies 2 (because there is inverse process)

Case of macroscopic systems
1 is satisfied due to low of large numbers. 2 is nontrivial condition.
Case of finite systems

To make $P_{A \rightarrow D}(\nu)$ delta function... set all transition rates zero except a particular $\nu = \nu^*$!
Consequence for finite systems

This condition is same for case of linear regime and case of finite $\Delta \mu$!

In finite systems, if Carnot efficiency is attainable in linear regime, it is attainable with any chemical potential difference $\Delta \mu$. 
Case of thermodynamic limit

\[ P_{A\rightarrow D}(\nu) = P_{A\rightarrow D;\nu} P_{A}^{eq}(\nu) \] has \( \delta \)-function type peak

transition rate with \( \nu \)

Equilibrium distribution at A

\[ P_{D\rightarrow A}(\nu) \propto e^{-\beta \Delta \mu \cdot V \nu} P_{A\rightarrow D}(\nu) \] in general has different peak position!
What is condition for not changing the position of peak?

(1) $\Delta \mu \to 0$ (linear regime)

$$P_{D \to A}(\nu) \propto e^{-\beta \Delta \mu \cdot \nu} P_{A \to D}(\nu) \text{ and } P_{A \to D}(\nu)$$

have peaks at the same position.

If equilibration condition is satisfied, macroscopic engines **attain the Carnot efficiency in linear regime!**
What is condition for not changing the position of peak?

(2) set transition rates zero except \( \nu = \nu^* \)

In this case, engines attain the Carnot efficiency with any \( \Delta \mu \).
What is condition for not changing the position of peak?

(3) Using indifferentiable point

Free energy or transition rate is indifferentiable

\[ f(\nu) \propto P_{A \rightarrow D;\nu} \cdot e^{-\beta F_A(\nu)} \]

(There indeed exists a model attaining the Carnot efficiency in this way)