Examining scenarios of thermalization: analytic model approach

Naoto Shiraishi (Keio University)


Financial support: SpringerNature Young Scientist Travel Grant
Outline

Motivation

Model 1 (embedded Hamiltonian)

Model 2 (Quantum emulation of CA)
Outline

Motivation

Model 1 (embedded Hamiltonian)

Model 2 (Quantum emulation of CA)
Thermalization of macroscopic system

Non-equilibrium state in macroscopic quantum system goes to a unique equilibrium state (Thermalization).

But some systems do not! (e.g., integrable systems)
Thermalization of macroscopic system

Non-equilibrium state in macroscopic quantum system goes to a unique equilibrium state (Thermalization).

But some systems do not! (e.g., integrable systems)

What determines whether a system thermalizes? What scenario generally explains thermalization?
Scenario 1: ETH scenario

Diagonal ETH (Eigenstate thermalization hypothesis)

All energy eigenstates are thermal

\[ \langle E_n | \hat{O} | E_n \rangle \simeq \text{Tr}[\hat{O} \rho_{MC}] \]  

(\hat{O}: macro observable)

Scenario 1: ETH scenario

**Diagonal ETH (Eigenstate thermalization hypothesis)**

All energy eigenstates are thermal

\[ \langle E_n | \hat{O} | E_n \rangle \approx \text{Tr}[\hat{O} \rho_{MC}] \] (\(\hat{O}\): macro observable)


It is proved that “ETH \(\rightarrow\) thermalization”.


Does ETH scenario explains all thermalization?
Scenario 2: large effective dimension scenario

**Effective dimension of $|\psi\rangle$:**

$$D_{\text{eff}} := \left( \sum_{n} |\langle E_n | \psi \rangle|^4 \right)^{-1}$$

Quantifier of how many energy eigenstates effectively cover $|\psi\rangle$. ($1 \leq D_{\text{eff}} \leq D = \text{dim}(\mathcal{H})$)
Scenario 2: large effective dimension scenario

Effective dimension of $|\psi\rangle$: $D_{\text{eff}} := \left( \sum_n |\langle E_n | \psi \rangle|^4 \right)^{-1}$

Quantifier of how many energy eigenstates effectively cover $|\psi\rangle$. ($1 \leq D_{\text{eff}} \leq D = \text{dim}(\mathcal{H})$)

It is proved that "initial state has a large $D_{\text{eff}}$ as

$D_{\text{eff}} / D = \text{poly} \left( \frac{1}{L} \right) \rightarrow \text{thermalization}"


Does large $D_{\text{eff}}$ scenario explains all thermalization?
Present situation

Numerical simulations imply that some non-integrable models satisfy both scenarios.

Present situation

Numerical simulations imply that some non-integrable models satisfy both scenarios.

No proof/disproof of these scenarios in concrete physical models of thermalization.

Why we have no proof/disproof?
Lack of analytically-accessible model of thermalization (Integrable systems do not thermalize)
Our goal

• Construct **analytically-accessible physical models** of thermalization in some sense.

• Prove/disprove two leading scenarios of thermalization in these models.
Remark: physical model

We use “physical model” in the sense that

- Hamiltonian is **local** (short-range)
- Hamiltonian is **translationally invariant**

For thermalization, we require

- There is **no local conserved quantity**
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Consider \( S=\frac{1}{2} \) spin chain with p.b.c. with length \( L \).
Construction of Hamiltonian

Consider \textbf{S}=1/2 spin chain with p.b.c. with length $L$. 

- site $i$:
- $S_{i}^{\text{tot,3}}$: total spin of these three spins
Construction of Hamiltonian

Consider a $S=1/2$ spin chain with p.b.c. with length $L$.

Projection operator:

$$P_i^{S=3/2} = \begin{cases} 
0 & S_{i,\text{tot},3} = 1/2 \\
1 & S_{i,\text{tot},3} = 3/2 
\end{cases}$$
Construction of Hamiltonian

Consider **S=1/2 spin chain** with p.b.c. with length $L$.

Projection operator: $(h_i$: Arbitrary Hamiltonian$)$

$$H = \sum_i P_i^{S=3/2} h_i P_i^{S=3/2} \quad (h_i$: Arbitrary Hamiltonian$)
Role of projection operators

\[ H = \sum_i p_{i}^{S=3/2} h_i p_{i}^{S=3/2} \]

where

\[ p_{i}^{S=3/2} = \begin{cases} 
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Role of projection operators

\[ H = \sum_i P_i^{S=3/2} h_i P_i^{S=3/2}, \quad P_i^{S=3/2} = \begin{cases} 0 : & S_{i}^{\text{tot},3} = 1/2 \\ 1 : & S_{i}^{\text{tot},3} = 3/2 \end{cases} \]

Diagram:
- Nodes labeled with indices \( i-1 \), \( i \), and \( i+1 \)
- Edge labeled with \( h_i \)
Role of projection operators

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Dimer states are energy eigenstates

\[ H = \sum_i P_i^{S=3/2} h_i P_i^{S=3/2}, \quad P_i^{S=3/2} = \begin{cases} 0: & S_i^{\text{tot},3} = 1/2 \\ 1: & S_i^{\text{tot},3} = 3/2 \end{cases} \]

\[ |\Psi^1_{\text{dimer}}\rangle := \prod_{n=1}^{L/2} |v_{2n-1,2n}\rangle, \quad |\Psi^2_{\text{dimer}}\rangle := \prod_{n=0}^{L/2-1} |v_{2n,2n+1}\rangle, \]

with spin singlet \[ |v_{i,j}\rangle := \frac{1}{\sqrt{2}}(|\uparrow\rangle_i |\downarrow\rangle_j - |\downarrow\rangle_i |\uparrow\rangle_j) \]

Dimer state

\[ \text{singlet} \quad \text{singlet} \]
Dimer states are energy eigenstates

\[ H = \sum_i P_i^{S=3/2} h_i P_i^{S=3/2}, \quad P_i^{S=3/2} = \begin{cases} 0 : & S_i^{\text{tot},3} = 1/2 \\ 1 : & S_i^{\text{tot},3} = 3/2 \end{cases} \]

Dimer state

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total spin = 0
Dimer states are energy eigenstates

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Dimer state

- total spin = 1/2
- total spin = 0
Dimer states are energy eigenstates

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Then, \( P_i^{S=3/2} |\Psi_{\text{dimer}}\rangle = 0 \) for all \( i \).

\[ |\Psi_{\text{dimer}}\rangle \] are eigenstates of \( H \) with energy zero!

Dimer state

\[ \text{total spin } = 1/2 \]

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- \[ h_i \] is completely arbitrary
  \[ \rightarrow \text{no local conserved quantity, thermalization.} \]
Dimer states are energy eigenstates

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\[ \rightarrow \left| \Psi_{\text{dimer}} \right\rangle \text{ are eigenstates of } H \text{ with energy zero!} \]

- \( h_i \) is completely arbitrary
  \[ \rightarrow \text{no local conserved quantity, thermalization.} \]
- \( h_i \) has no relation to \( \left| \Psi_{\text{dimer}} \right\rangle \)
  \[ \rightarrow \left| \Psi_{\text{dimer}} \right\rangle \text{ are non-thermal energy eigenstates.} \]
Counterexample to diagonal ETH

\[ \frac{\langle E_n | A | E_n \rangle}{L} \]

\[ \langle E_n \rangle \]
Counterexample to diagonal ETH

\[ \frac{\langle E_n | A | E_n \rangle}{L} \]

thermal average
Counterexample to diagonal ETH

\[ \frac{\langle E_n | A | E_n \rangle}{L} \]

Two dimer states are indeed non-thermal!
Thermalization without ETH

Do these two non-thermal eigenstates disturb thermalization? → No!
Thermalization without ETH

Do these two non-thermal eigenstates disturb thermalization? → **No!**
Thermalization without ETH

Do these two non-thermal eigenstates disturb thermalization? → No!

not thermalize

thermalize

singlet

triplet!

singlet
Thermalization without ETH

Do these two non-thermal eigenstates disturb thermalization? → No!

→ All preparable initial states thermalize.

(ETH scenario is not applicable to this thermalization)
Generalization: Method of embedding

\{ |\psi\rangle \}: target states we want to embed

\( P_i \): local projection operators

\( P_i |\psi\rangle = 0 \)
Generalization: Method of embedding

\{ |\psi\rangle \}: target states we want to embed

\( P_i \): local projection operators

\[ P_i |\psi\rangle = 0 \]

Then, the Hamiltonian with arbitrary \( h_i \)

\[ H = \sum_i P_i h_i P_i \]

has the target states \( \{ |\psi\rangle \} \) as its thermal eigenstates.
Generalization: Method of embedding

\{ |\psi\rangle \}: target states we want to embed

\[ P_i: \text{local projection operators} \]

\[ P_i |\psi\rangle = 0 \]

Then, the Hamiltonian with arbitrary \( h_i \)

\[ H = \sum_i P_i h_i P_i \]

has the target states \{ |\psi\rangle \} as its thermal eigenstates.

Any MPS (matrix-product state) can be target state! (ex: AKLT state, Schrodinger cat state, ...)
Dynamics (prethermalization)

Embedding large subspace
→ prethermalization (two-step relaxation) occurs!

ex) spin-1 chain
{\{|\psi\rangle\}}: all spins are ±1

Related to glassy dynamics.

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  • Classical cellular automata (CA)
  • Construction of quantum emulating system
  • Analytic results
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1d Classical reversible CA (cellular automata)

Ex: Rule214

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n step

\[ \cdot \cdot \cdot \]

\[ \begin{array}{cccccc}
1 & 0 & 0 & 1 & 1 & 0 \\
\end{array} \cdot \cdot \cdot \]

n+1 step

\[ \cdot \cdot \cdot \]

\[ \cdot \cdot \cdot \]

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n step

n+1 step
1d Classical reversible CA
(cellular automata)

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n-1 step

n step

n+1 step
1d Classical reversible CA (cellular automata)

Ex: Rule214

Input: \(\begin{array}{cccccc}
1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\)

Output: \(\begin{array}{cccccc}
1 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\)
1d Classical reversible CA (cellular automata)

Ex: Rule 214

Input: 1 1 1 1 0 1 0 0

Output: 1 0 0 1 1 1 0

\[ \oplus \equiv (\text{mod} \ 2) \]
State space decomposition into cycles

\[
\begin{array}{ccccccc}
& n-1 \text{ step} & \cdots & 0 & 1 & 1 & 1 & 0 & 0 & \cdots \\
n \text{ step} & \cdots & 1 & 0 & 0 & 1 & 1 & 0 & \cdots \\
n+1 \text{ step} & \cdots & 1 & 0 & 1 & 0 & 1 & 1 & \cdots \\
\end{array}
\]
State space decomposition into cycles

- n-1 step: 
  
  \[ \begin{array}{ccccccc}
  \cdot & \cdot & 0 & 1 & 1 & 1 & 0 & 0 \\
  \cdot & \cdot & 1 & 0 & 0 & 1 & 1 & 0 \\
  \cdot & \cdot & 1 & 0 & 1 & 0 & 1 & 1 \\
  \end{array} \]

- n step: 
  
  \[ \begin{array}{ccccccc}
  0 & 1 & 1 & 1 & 0 & 0 \\
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  1 & 0 & 1 & 0 & 1 & 1 \\
  \end{array} \]

- n+1 step: 
  
  \[ \begin{array}{ccccccc}
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  1 & 0 & 1 & 0 & 1 & 1 \\
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(Generalized) state
State space decomposition into cycles

State space is decomposed into some cycles.

\[ u_k : \text{length of cycle } k \]
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Model 2 (Quantum emulation of CA)
  • Classical cellular automata (CA)
  • Construction of quantum emulating system
  • Analytic results
Structure of emulating quantum system

single fermion: 
$L$ possible states

head system

CA system

$2L$ spins
$2^{2L}$ possible states
Correspondence between CA and quantum system

<table>
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<tr>
<th>classical CA</th>
<th>n-1 step</th>
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The figure shows the correspondence between classical and quantum systems, with the evolution of states at each step.
Correspondence between CA and quantum system

| classical CA | n-1 step | · · · | 0 | 1 | 1 | 1 | 1 | 0 | 0 | · · · |
|             | n step   | · · · | 1 | 0 | 0 | 1 | 1 | 1 | 0 | · · · |
|             | n+1 step | · · · | 1 | 0 | 1 | 0 | 1 | 1 | 0 | · · · |

| quantum     | · · · | · · · | 1 | 0 | 1 | 0 | 1 | 1 | 0 | · · · |
|             | · · · | · · · | 1 | 0 | 0 | 1 | 1 | 1 | 0 | · · · |
Correspondence between CA and quantum system

| classical CA | n-1 step | 0 | 1 | 1 | 1 | 0 | 0 | ... |
| n step       | ...      | 1 | 0 | 0 | 1 | 1 | 0 | ... |
| n+1 step     | ...      | 1 | 0 | 1 | 0 | 1 | 1 | ... |

| quantum      | ...      | 1 | 0 | 1 | 1 | 0 | 0 | ... |
| ...          | ...      | 1 | 0 | 0 | 1 | 1 | 0 | ... |
| ...          | ...      | 1 | 0 | 1 | 1 | 0 | 0 | ... |
Correspondence between CA and quantum system

classical CA

n-1 step • • • 0 1 1 1 1 0 0 • • •
n step • • • 1 0 0 1 1 1 0 • • •
n+1 step • • • 1 0 • • •

quantum • • • • • •

1 0 1 1 1 0 0 • • •
1 0 0 1 1 1 0 • • •
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Correspondence between CA and quantum system
local Hamiltonian

\[ h_i = \sum_{p,q,r,s \in \{0,1\}} \left| \begin{array}{ccc} p & r & * \\ * & X & s \end{array} \right\rangle \left\langle \begin{array}{ccc} p & q & * \\ * & r & s \end{array} \right| \otimes c_{i+1}^\dagger c_i + \text{c.c.} \]
local Hamiltonian

\[ h_i = \sum_{p,q,r,s \in \{0,1\}} \left| \begin{array}{cc} p & r \\ * & X \\ * & r \\ * & s \end{array} \right\rangle \left\langle \begin{array}{cc} p & q \\ * & X \\ * & r \\ * & s \end{array} \right|_i \otimes c_{i+1}^\dagger c_i + \text{c.c.} \]

jump of fermion
local Hamiltonian

\[ h_i = \sum_{p,q,r,s \in \{0,1\}} \begin{pmatrix} p \mid r \end{pmatrix} \begin{pmatrix} s \end{pmatrix} \begin{pmatrix} p \mid q \end{pmatrix} \begin{pmatrix} r \mid s \end{pmatrix}_i \otimes c_{i+1}^\dagger c_i + \text{c.c.} \]

jump of fermion
Single rotation of fermion updates state of CA system

n-1 step
n step
n+1 step
Single rotation of fermion updates state of CA system

n-1 step

n step

n+1 step
Single rotation of fermion updates
state of CA system

n-1 step
n step
n+1 step
Single rotation of fermion updates state of CA system
Single rotation of fermion updates state of CA system

We set proper boundary condition to keep consistency.
Structure of state space

Free fermion
Structure of state space

Free fermion

Our quantum system

cycle length $u_k$
Explicit form of energy eigenstate

\[ |E_{k,m}\rangle = \frac{1}{\sqrt{u_kL}} \sum_{n=1}^{u_k} \sum_{i=1}^{L} e^{-\frac{2\pi im(nL+i)}{u_k L}} |X_{k,i}^n\rangle \otimes |i\rangle \]
Explicit form of energy eigenstate

\[ |E_{k,m}\rangle = \frac{1}{\sqrt{u_k L}} \sum_{n=1}^{u_k} \sum_{i=1}^{L} e^{-\frac{2\pi im(nL+i)}{u_k L}} |X_{k,i}^n\rangle \otimes |i\rangle \]
This talk: single head particle

Generalization: many interacting head particles (fermion/hard-core boson)

This model is still formally solvable.
(We need some modification on the CA)
Outline

Motivation

Model 1 (embedded Hamiltonian)

Model 2 (Quantum emulation of CA)

- Classical cellular automata (CA)
- Construction of quantum emulating system
- Analytic results
Properties of classical CA (1)

chaos

Observed fact by numerical simulation (1)

Some CA are **chaotic**.

Time evolution of R214

\[
0 = \begin{array}{c}
\boxempty
\end{array}
\]

\[
1 = \begin{array}{c}
\blacklozenge
\end{array}
\]

Properties of classical CA (2)

single cycle does not cover state space

Observed fact by numerical simulation (2)

Single cycle covers **exponentially small fraction** of state space.

\( u_k / 2^{2L} = O(e^{-L}) \) for all \( k \)

Assumption on classical CA

We assume that there exists classical CA such that

1. chaotic

2. \( \frac{u_k}{2^{2L}} = O(e^{-L}) \) for all \( k \)

(Note: These conditions can be written in mathematical form)

We have no proof, but numerical simulations strongly support existence of such CA.
We assume that there exists classical CA such that

1. chaotic

2. \( u_k / 2^{2L} = O(e^{-L}) \) for all \( k \)

(Note: These conditions can be written in mathematical form)

We have no proof, but numerical simulations strongly support existence of such CA.

In the following, we emulate this chaotic CA.
Thermalization via emulation of classical chaotic CA

Result 1: This quantum system thermalizes as long as we observe quantities in CA system.

Thermalization: partial trace of state to small region coincides with that of microcanonical ensemble.
**Scenario 1: ETH scenario**

**Diagonal ETH (Eigenstate thermalization hypothesis)**

All energy states are thermal

\[
\langle E_n | \hat{O} | E_n \rangle \approx \text{Tr}[\hat{O} \rho_{MC}] \quad (\hat{O}: \text{macro observable})
\]


It is proved that “ETH → thermalization”.

Does ETH scenario explains this thermalization?
This thermalization is not explained by ETH scenario

Classical CA

\[
\begin{array}{cccccc}
0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{array}
\]

Spatial periodicity of initial state is kept in time evolution.
This thermalization is not explained by ETH scenario

Classical CA

Spatial periodicity of initial state is kept in time evolution.
This thermalization is not explained by ETH scenario.

Spatial periodicity of initial state is kept in time evolution.

Result 2: Corresponding energy eigenstate has short spatial period, and thus is nonthermal. → ETH scenario is not applicable!
Coexistence of violation of ETH and thermalization

Initial state with complete spatial period → not thermalize

\[ \begin{array}{ccccccccccc}
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & \ldots
\end{array} \]
Coexistence of violation of ETH and thermalization

Initial state with complete spatial period
→ not thermalize

\[ \cdots 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ \cdots \]

Initial state with spatial period with single defect
→ thermalizes!

\[ \cdots 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 1 \ 0 \ 0 \ \cdots \]
Coexistence of violation of ETH and thermalization

Initial state with complete spatial period
→ not thermalize

\[ \cdots 0 1 0 1 0 1 0 1 0 1 0 1 \cdots \]

Initial state with spatial period with single defect
→ thermalizes!

\[ \cdots 0 1 0 1 0 1 0 0 0 1 0 1 \cdots \]
Scenario 2: large effective dimension scenario

Effective dimension of $|\psi\rangle$: $D_{\text{eff}} := \left( \sum_n |\langle E_n |\psi\rangle|^4 \right)^{-1}$

Quantifier of how many energy eigenstates effectively cover $|\psi\rangle$. ($1 \leq D_{\text{eff}} \leq D = \text{dim}(\mathcal{H})$)

It is proved that "initial state has a large $D_{\text{eff}}$ as $D_{\text{eff}}/D = \text{poly}\left(\frac{1}{L}\right) \rightarrow \text{thermalization}"

Does large $D_{\text{eff}}$ scenario explains this thermalization?
This thermalization is not explained by large $D_{\text{eff}}$ scenario

Consider an initial state as $|X^n_{k,i}\rangle \otimes |i\rangle$
This thermalization is not explained by large $D_{\text{eff}}$ scenario

Consider an initial state as $|X_{k,i}^n \otimes |i\rangle$

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effective dimension $= \text{cycle length } u_k$

**Result 3:** Cycle length is exponentially small $= \text{effective dimension is not large!}$

$\rightarrow$ Large $D_{\text{eff}}$ scenario is not applicable.
Summary (tentative)

First model (embedded Hamiltonian)
Idea: target state is embedded as energy eigenstate. **ETH scenario fails to explain this thermalization.**

Second model (quantum emulation of classical CA)
Idea: chaotic behavior is directly emulated. With some assumptions on classical chaos, **neither scenarios explain this thermalization.**

We need another scenario of thermalization.
Another possible scenario

Specific physical models → not easy → ETH → proved

not easy → large $D_{eff}$ → proved

no local conserved quantity → thermalization
Another possible scenario

specific physical models

not easy

not easy

proved! (e.g., XYZ+h chain)

proved

proved

ETH

large $D_{\text{eff}}$

no local conserved quantity

thermalization

(N. Shiraishi, arXiv:1803.02637)
Another possible scenario

Specific physical models

not easy

not easy

large $D_{\text{eff}}$

proved

proved!

proved! (e.g., XYZ+h chain)

no local conserved quantity

reativeation

(N. Shiraishi, arXiv:1803.02637)
Difference between with and without local conserved quantity

With local conserved quantity (ex: integrable model)

Subspace spanned by thermal eigenstate

Subspace spanned by non-thermal eigenstate

Without local conserved quantity (ex: our models)

Initial state we want to set
Difference between with and without local conserved quantity

With local conserved quantity (ex: integrable model)
- subspace spanned by thermal eigenstate
- initial state we want to set thermal (local) noise

Without local conserved quantity (ex: our models)
- subspace spanned by non-thermal eigenstate
Difference between with and without local conserved quantity

- Subspace spanned by thermal eigenstate
- Subspace spanned by non-thermal eigenstate

With local conserved quantity (ex: integrable model)

Without local conserved quantity (ex: our models)
Conjecture

Non-thermalizing initial state is stably preparable (against thermal noise) only if non-thermal subspace is characterized by local conserved quantity.
Conjecture

Non-thermalizing initial state is **stably preparable (against thermal noise)** only if non-thermal subspace is characterized by **local conserved quantity**.

Presence/absence of local conserved quantity determines whether system thermalizes.
Summary

First model (embedded Hamiltonian)
Idea: target state is embedded as energy eigenstate. **ETH scenario fails to explain this thermalization.**

Second model (quantum emulation of classical CA)
Idea: chaotic behavior is directly emulated. With some assumptions on classical chaos, **neither scenarios explain this thermalization.**

We need another scenario of thermalization.
Known properties of ETH

- **ETH is sufficient for thermalization.**

- **Many complex (non-integrable, non-localized) systems satisfy ETH.**
  

- **All known counterexamples to ETH are only non-thermalizing systems.**
  
  - Integrable systems (with local conserved quantities)
  - Localized systems
Diagonal ETH in each sector and whole Hilbert space

(d1): diagonal ETH in the whole Hilbert space.
(d2): diagonal ETH in each sector.

Numerical simulation before our result
With local conserved quantities $\rightarrow$ only (d2) holds.

With only nonlocal conserved quantity $\rightarrow$ (d1) holds.
(ex: translational symmetry, mirror symmetry, ...)

Thermalization is ensured only by (d1)

(d1): diagonal ETH in the whole Hilbert space.
(d2): diagonal ETH in each sector.

(d1) holds $\Rightarrow$ System thermalizes.
Diagonal ensemble = microcanonical

(d2) holds $\Rightarrow$ System not necessarily thermalizes.
(Diagonal ensemble may differ from microcanonical ensemble).
Nonlocal conserved quantity?

In our model, nonlocal conserved quantity is

\[ P = |\Psi^1_{\text{dimer}}\rangle \langle \Psi^1_{\text{dimer}}| + |\Psi^2_{\text{dimer}}\rangle \langle \Psi^2_{\text{dimer}}| \]

This is just the projection operator to non-thermal energy eigenstates!

They are conserved, but we in general not take into account projector to energy eigenstates as a conserved quantity (see next slide).
Why we should NOT consider projection operator as nonlocal conserved quantity

Using the following non-local conserved quantity,

\[ P = \sum_{|E_i\rangle: \text{nonthermal}} |E_i\rangle \langle E_i| \]

all nonthermal energy eigenstates are removed, and all the energy eigenstates in the remaining subspace are thermal by definition!

non-thermal states:

We can remove them by \( P \)
Relaxation and off-diagonal ETH

Relaxation is ensured if
(i) off-diagonal ETH holds, or
(ii) $D_{\text{eff}}$ diverges.


Diverging effective dimension (ii) is shown for exponentially-decaying states